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ORIGINAL PAPER

Weighted pseudo almost automorphic classical solutions and optimal mild solutions for fractional differential equations and application in fractional reaction–diffusion equations

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Abstract In this paper, we are concerned with a class of fractional differential equations given by

$$D_t^{\alpha} x(t) = Ax(t) + f(t, x(t)).$$

Our main results concern the existence, uniqueness of weighted pseudo-almost automorphic classical solutions and optimal mild solutions. Moreover, as example and applications, we study the weighted pseudo-almost automorphic classical solutions and optimal mild solutions for a fractional reaction–diffusion equation to illustrate the practical usefulness of the analytical results that we establish in the paper.

Keywords Optimal mild solution · Weighted pseudo-almost automorphic classical solution · Fractional differential equation · Fractional reaction–diffusion equation · Existence and uniqueness

1 Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. In recent years, it has turned out that many phenomena

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in engineering, physics, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. Fractional derivatives are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering. Fractional derivative is becoming a popular tool to model various physical phenomena and to describe the dynamical characteristics of the physical system better than the standard integer order derivatives. For example, the reaction–diffusion problem is better described by fractional differential equation than the classical diffusion equation when there is a presence of anomalous diffusion of particles [1,2]. There are some probabilistic interpretations of the fractional derivatives that better suits in the control theory as well [3,4].

The fractional calculus was first anticipated by Leibnitz, was one of the founders of standard calculus, in a letter written in 1695. This calculus involves different definitions of the fractional operators as well as the Riemann-Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald-Letnikov fractional derivative [5]. The fractional calculus has gained considerable importance during the past decades mainly due to its applications in diverse fields of science and engineering. One observes that fractional order can be complex in viewpoint of pure mathematics and they have recently proved to be valuable in various fields of science and engineering. Indeed, one can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, biology and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [6,7] or to model activator-inhibitor dynamics with anomalous diffusion [8]. For details, see [9–11] and the references therein.

Meanwhile due to their applications in fields of science where fractional differential equations have attracted increasing attention, and notable contributions have been made to the applications of fractional differential equations. These equations are increasingly applied to efficient model problems in research areas as diverse as machanical systems, dynamical systems, control, chaos,continuous time random walks, anomalous diffusive and sub-diffusive systems, wave propagation and so on. Mathematical modelling of complex processes is a major challenge for contemporary scientist. In contrast to simple classical systems, where the theory of integer order differential equations is sufficient to describe their dynamics, fractional derivatives provide an excellent and an efficient instrument for the description of memory and hereditary properties of various complex materials and systems.

The reaction–diffusion equations arise naturally as description models of many evolution problems in the real world, as in chemistry (Slepchenko et al. [12]; Vidal and Pascault [13]), biology (Murray [14]), etc. Mathematically, the reaction–diffusion systems take the form of semilinear parabolic partial differential equations. Usually, in real world applications, the reaction term describes the birth-death or reaction occurring inside the habitat or reactor. The diffusion term models the movement of many individuals in an environment or media. The individuals can be very small particles in physics, bacteria, molecules, or cells, or very large objects such as animals, plants. As is well known, complex behavior is peculiarity of systems modeled by reaction–diffusion equations and the Belousov–Zhabotinskii reaction (Muller et al. [15]; Winfree [16]) provides a classic example. In recent years, reaction–diffusion equations have been widely studied and applied in the fields of logistic population growth, flame propagation, euro physiology, autocatalytic chemical reactions, branching Brownian motion processes, and nuclear reactor theory. For example reaction–diffusion equations are commonly applied to model the growth and spreading of biological species (Murray [14]), and been used as a basis for a wide variety of models, for the special spread of gene in population and for chemical wave propagation.

The diffusion of two or more chemicals at unequal rates over a surface react with one another in order to form stable patterns is represented by reaction diffusion equation. The nature of the diffusion is characterized by temporal scaling of the mean square displacement $< r^2(t) > \propto t^\alpha$. For standard diffusion $\alpha = 1$, whereas in anomalous subdiffusion $\alpha < 1$ and in anomalous superdiffusion $\alpha > 1$. These situations are called anomalous diffusion [17,18]. Subdiffusion typically arises in cases where there are spatial or temporal constraints such as occur in fractured and porous media and fractal lattices. Superdiffusion may occur in chaotic or turbulent processes through enhanced transport of particles. The review paper by Klafter et al. [19] provides numerous references to physical phenomena in which anomalous diffusion occurs. One popular model for anomalous diffusion is the fractional diffusion equation, where the usual second derivative in space is replaced by a fractional derivative of order $0 < \alpha < 2$ [1,20]. Standard diffusion is represented by classical diffusion equations and subdiffusion and superdiffusion are represented by fractional diffusion equations. Mainardi and Mainardi et al. [21] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α, and they proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2. In [22], Oldham and Spanier discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and half order derivative in time. The fundamental solutions of the Cauchy problems associated to these generalized diffusion equations ($0 < \alpha \leq 2$) are studied in [21,23]. Recent research indicates that the classical diffusion equation is inadequate to model many real situations, where a particle plume spreads faster than the classical model predicts, and may exhibit significant asymmetry. Solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

A fractional reaction–diffusion equation is derived from a continuous time random walk model when the transport is dispersive. The exit from the encounter distance, which is described by the algebraic waiting time distribution of jump motion, interferes with the reaction at the encounter distance. Therefore, the reaction term has a memory effect. The derived equation is applied to the geminate recombination problem. The recombination is shown to depend on the intrinsic reaction rate, in contrast with the results of Sung et al. [24], which were obtained from the fractional reaction–diffusion equation where the diffusion term has a memory effect but the reaction term does not. In the recent years, there has been a great deal of interest in fractional reaction–diffusion systems which from one side exhibit selforganization phenomena and from the other side introduce a new parameter to these systems, which is a fractional derivative index, and it gives a great degree of freedom for diversity of selforganization phenomena and new nonlinear effects depending on the order of time-space fractional derivatives. From a mathematics point of view they also offer a rich and promising area of research. The most important advantage of using fractional differential equations is their nonlocal property. This indicates that the next state of a system depends not only upon its current state but also upon all of its previous states. In recent years, the fractional reaction–diffusion equation has received the applications in systems biology [25,26], chemistry, and biochemistry applications [27]. A strong motivation for studying and investigating the solution and the properties for fractional diffusion equations comes from the fact that they describe efficiently anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [17,28–33] and references therein), fractional random walk, etc.

In the earlier sixties, Bochner introduced the concept of almost automorphic function in his papers [34–36] in relation to some aspects of differential geometry. The notion of almost automorphic function was introduced to avoid some assumptions of uniform convergence that arise when using almost periodic function, it is an important generalization of the classical almost periodic function which is one of the most attractive topics in the qualitative theory of differential equations because of its significance and applications in physics, mathematical biology, control theory, and other related fields. In the last several decades, the basic theories on the almost automorphic functions have been well developed [37–39], and been applied successfully to the investigation of almost automorphic dynamics produced by many different kinds of differential equations [40–42]. As a result, several concepts were introduced as generalizations or restrictions of almost automorphy, such as asymptotic almost automorphy, pseudo almost automorphy, weighted pseudo almost automorphy (see, for example [43–45]).

The concept of pseudo almost automorphy has recently been introduced in the literature by Liang, Xiao and Zhang [45,46], as a powerful generalization of both the notion of almost automorphy due to Bochner [34–36] and that of pseudo almost periodicity due to Zhang [47–49]. Since then, the existence of pseudo-almost automorphic solutions to differential equations, partial differential equations, and functional differential equations has been of a great interest to several authors and hence generated various contributions [50–52]. For more on this concept and related topics, see, e.g., [53–55] and references therein.

In 2009, Blot, Mophou, N'Guérékata, Pennequin [44] introduced the concept of weighted pseudo almost automorphic function, which is a generalization of the classical almost automorphic functions of Bochner [34–36], the asymptotically almost automorphic functions of N'Guérékata [43] as well as the pseudo almost automorphic functions of Liang, Xiao and Zhang [45,46]. Recently, weighted pseudo almost automorphic functions are widely investigated and used in the study of differential equations. Many basic properties and applications to several classes of differential equations were established, see, for example, [56–58] and references therein. On the other hand, the properties of weighted pseudo almost automorphic functions are more complicated and changeable than the almost automorphic functions and the pseudo almost automorphic functions because the influence of the weight ρ is very strong sometimes, and the theory of weighted pseudo almost automorphic functions is worthy to be studied deeply by new ideas.

In this paper, we study some sufficient conditions for the existence, uniqueness of optimal mild solutions and weighted pseudo-almost automorphic classical solutions to the following fractional differential equations

$$\mathrm{D}_t^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in I := [t_0, T], \quad x(t_0) = x_0 \tag{1.1}$$

where $\mathrm{D}_t^\alpha x(t)$ is the standard Riemann-Liouville fractional derivative, $0 < \alpha \leq 1$, A is the infinitesimal generator of a analytic semigroup $\{Q(t)\}_{t\geq 0}$ in X, $f : \mathbb{R}\times X_q \to X$ satisfies suitable conditions. Moreover, the weighted pseudo-almost automorphic classical solutions and optimal mild solutions for a fractional reaction–diffusion equation, which is illustrated by example, are in good agreement with the theoretical analysis.

The Eq. (1.1) for a particular case in which $\alpha = 1$ has been considered by Bahuguna and Srvastavai [59]. The existence of a unique mild solution to Eq. (1.1) with $\alpha = 1$ is assured under the conditions that A is the infinitesimal generator of a compact semigroup in X, $f(t, x)$ is continuous in both the variables and uniformly locally Lipschitz continuous in x. If the Lipschitz continuity of f in x is dropped, then the existence of a mild solution is no more guaranteed, Examples, in which $A = 0$, f is continuous and the differential equations do not have solutions are given in Dieudonne [60] and Yorke [61]. Very recently, Balachandran and Park [62] have studied the existence and uniqueness of solutions to Eq. (1.1) with nonlocal initial conditions. The main approach used in [62] is the Krasnoselskii's fixed point theorem. For more information in this fields, see [10,62–66] and the references therein.

The rest of this paper is organized as follows. In Sect. 2, some concepts and the relating notations are introduced. In Sect. 3, some criteria ensuring the existence and uniqueness of solutions are presented. In Sect. 4, the existence and uniqueness of optimal mild solutions were proved. In Sect. 5, the existence and uniqueness of weighted pseudo-almost automorphic classical solutions were proved. Finally, in Sect. 6, as example and applications, we study the weighted pseudo-almost automorphic classical solutions and optimal mild solutions for a fractional reaction–diffusion equation to illustrate the practical usefulness of the analytical results that we establish in the paper.

2 Preliminaries

From now on, let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces, $L(X)$ is the Banach space of all linear and bounded operators on X. $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R}\times Y, X)$) is the space of all X-valued bounded continuous functions (resp., jointly bounded continuous functions $f : \mathbb{R}\times Y \to X$). Furthermore, $C(\mathbb{R}, X)$ (resp., $C(\mathbb{R}\times Y, X)$) denotes the class of continuous functions from $\mathbb{R}$ into X (resp., jointly continuous functions $f : \mathbb{R}\times Y \to X$). For a linear operator A with domain $D(A)$, denote by $\mathbb{R}(A)$, $\rho(A)$ the range and the resolvent set of A.

In the following, we firstly give a brief outline of the theory of fractional powers as developed in [67]. Let A be the infinitesimal generator of an analytic semigroup

$\{Q(t)\}_{t\geq 0}$ in Banach space X and $0 \in \rho(A)$. For $q > 0$, define the fractional power A^{-q} by

$$A^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} Q(t) \mathrm{d}t.$$

For $0 < q \leq 1$, A^q is a closed linear operator whose domain $D(A^q) \supset D(A)$ is dense in X. The closedness of A^q implies that $D(A^q)$ endowed with the graph norm

$$\|x\|_{D(A)} = \|x\| + \|A^q x\|, \quad x \in D(A^q),$$

is a Banach space. Clearly $A^q = (A^{-q})^{-1}$, because A^{-q} is one to one. Since $0 \in \rho(A)$, A^q is invertible, and its graph norm is equivalent to the norm $\|x\|_q = \|A^q x\|$. Thus $D(A^q)$ equipped with the norm $\|\cdot\|_q$ is a Banach space which denotes by X_q.

Lemma 2.1 [67] *Let A be the infinitesimal generator of an analytic semigroup $\{Q(t)\}_{t\geq 0}$. If $0 \in \rho(A)$, then*

(1) $Q(t) : X \to D(A^q)$ for every $t > 0$ and $q \geq 0$,
(2) For every $x \in D(A^q)$, one has $Q(t)A^q x = A^q Q(t)x$,
(3) For every $t > 0$, the operator $A^q Q(t)$ is bounded and $\|A^q Q(t)\|_{L(X)} \leq M_q t^q e^{-\delta t}$,
(4) For $0 < q \leq 1$ and $x \in D(A^q)$, one has $\|Q(t)x - x\| \leq C_q t^q \|A^q x\|$.

The concept of pseudo-almost automorphic function is a natural generalization of that of almost automorphic function, and the concept of almost automorphic functions was first created by Bochner. Since then, those functions have been widely studied and developed. Now, let us recall some basic definitions and results on almost automorphic functions and pseudo almost automorphic functions.

Definition 2.1 (Bochner [36]) A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n=1}^\infty$, one can extract a subsequence $\{s_n\}_{n=1}^\infty$ such that

$$g(t) = \lim_{n\to\infty} f(t + s_n),$$

is well defined in $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t),$$

for each $t \in \mathbb{R}$.

Denote by $AA(\mathbb{R}, X)$ the set of all such functions.

Remark 2.1 The function g in Definition 2.1 is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous (cf., e.g., [68], Theorem 2.6). If the convergence in Definition 2.1 is uniform in $t \in \mathbb{R}$, then f is almost periodic. A classical example of almost automorphic function (not almost periodic) is (cf. [69])

$$f(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right), \quad t \in \mathbb{R}.$$

Definition 2.2 [36] A continuous function $f : \mathbb{R} \times Y \times Y \to X$ is said to be almost automorphic if $f(t, x, y)$ is automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where K is any bounded subset of $Y \times Y$. That is to say, for every sequence of real numbers $\{s'_n\}_{n=1}^{\infty}$, one can extract a subsequence $\{s_n\}_{n=1}^{\infty}$ such that

$$g(t, x, y) = \lim_{n\to\infty} f(t + s_n, x, y),$$

is well defined in $t \in \mathbb{R},\ (x, y) \in K$, and

$$\lim_{n\to\infty} g(t - s_n, x, y) = f(t, x, y),$$

for each $t \in \mathbb{R},\ (x, y) \in K$.

The collection of those functions is denoted by $AA(\mathbb{R} \times Y \times Y, X)$.

Lemma 2.2 [37] $(AA(\mathbb{R}, X), \|\cdot\|_{AA(\mathbb{R},X)})$ *is a Banach space with the supremum norm given by*

$$\|f\|_{AA(\mathbb{R},X)} = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

Lemma 2.3 [37] *Assume that $f : \mathbb{R} \to X$ is almost automorphic, then f is bounded.*

Let

$$PAP_0(X) := \left\{\varphi \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\varphi(t)\|dt = 0\right\},$$

$$PAP_0(Y \times Y, X) := \Big\{\varphi \in C(\mathbb{R} \times Y \times Y, X) : \varphi(\cdot, x, y) \ \ \textit{is bounded for each}$$

$$\times (x, y) \in Y \times Y \ \ \textit{and} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\varphi(t, x, y)\|dt = 0$$

$$\times \textit{uniformly in compact subset of} \ \ Y \times Y\Big\}.$$

Definition 2.3 [45] A continuous function $f : \mathbb{R} \to X$ is said to be pseudo almost automorphic if it can be decomposed as

$$f = g + \varphi,$$

where $g \in AA(\mathbb{R}, X)$ and $\varphi \in PAA_0(\mathbb{R}, X)$.

Denote by $PAA(\mathbb{R}, X)$ the set of all such functions.

Definition 2.4 [45] A continuous function $f : \mathbb{R} \times Y \times Y \to X$ is said to be pseudo almost automorphic if it can be decomposed as

$$f = g + \varphi,$$

where $g \in AA(\mathbb{R} \times Y \times Y, X)$ and $\varphi \in PAA_0(\mathbb{R} \times Y \times Y, X)$.

Denote by $PAA(\mathbb{R} \times Y \times Y, X)$ the set of all such functions.

Blot, Mophou, N'Guérékata, Pennequin [44] introduced the concept of weighted pseudo almost automorphic function, which is a generalization of the classical almost automorphic functions of Bochner [34–36], the asymptotically almost automorphic functions of N'Guérékata [43] as well as the pseudo almost automorphic functions of Liang, Xiao and Zhang [45,46], and the author gave some properties of the space of weighted pseudo almost automorphic functions such as the completeness and the composition theorem. Let us explain the meaning of this notion.

Using the same setting as in [44], let U be the collection of all piecewise continuous functions $\rho : \mathbb{R} \to [0, \infty)$ satisfying $\rho \in L_{\text{loc}}(\mathbb{R})$. If $\rho \in U$ and $T > 0$, define

$$m(T, \rho) := \int_{-T}^{T} \rho(x)\mathrm{d}x.$$

As in the particular case when $\rho(x) = 1$ for each $x \in \mathbb{R}$, we are exclusively interested in those weighted ρ, for which, $m(T, \rho) \to \infty$ as $T \to \infty$. Throughout the rest of the work, the notations U_∞ and U_B stands for the sets of weight functions

$$U_\infty := \{\rho \in U : \lim_{T\to\infty} m(T, \rho) = \infty\},$$
$$U_B := \{\rho \in U_\infty \text{ is bounded} : \liminf_{x\to\infty} \rho(x) > 0\}.$$

Let $\rho \in U_\infty$, define the "weighted ergodic" space

$$PAP_0(X, \rho) := \left\{\varphi \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \|\varphi(t)\|\rho(t)\mathrm{d}t = 0\right\},$$

$$PAP_0(Y, X, \rho) := \Big\{\varphi \in C(\mathbb{R} \times Y, X) : \varphi(\cdot, y) \quad \text{is bounded for each} \quad y \in Y \quad \text{and}$$
$$\times \lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \|\varphi(t, y)\|\rho(t)\mathrm{d}t = 0$$
$$\times \text{uniformly in compact subset of} \quad Y\Big\}.$$

Definition 2.5 [44] A continuous function $f : \mathbb{R} \to X$ is said to be weighted pseudo almost automorphic if it can be decomposed as

$$f = g + \varphi,$$

where $g \in AA(\mathbb{R}, X, \rho)$ and $\varphi \in PAA_0(\mathbb{R}, X, \rho)$.

Denote by $WPAA(\mathbb{R}, X, \rho)$ the set of all such functions.

Definition 2.6 [44] A continuous function $f : \mathbb{R} \times Y \times Y \to X$ is said to be weighted pseudo almost automorphic if it can be decomposed as

$$f = g + \varphi,$$

where $g \in AA(\mathbb{R} \times Y \times Y, X, \rho)$ and $\varphi \in PAA_0(\mathbb{R} \times Y \times Y, X, \rho)$.

Denote by $WPAA(\mathbb{R} \times Y \times Y, X, \rho)$ the set of all such functions.

Remark 2.2 [44] When $\rho = 1$, we obtain the standard spaces $PAA(\mathbb{R}, X)$ and $PAA(\mathbb{R} \times Y \times Y, X)$.

Let V_∞ be the collection of all continuous weights $\rho : \mathbb{R} \to (0, \infty)$ so that for every $\tau \in \mathbb{R}$

$$\limsup_{s \to \infty} \rho(s + \tau)/\rho(s) < \infty, \quad \limsup_{T \to \infty} m(t + \tau, \rho)/m(t, \rho) < \infty.$$

Remark 2.3 [44] If $\rho \in V_\infty$, then the space $PAP(X, \rho)$ is translation invariant.

Lemma 2.4 [44] *The decomposition of a weighted pseudo almost automorphic function is unique for any $\rho \in U_B$.*

Lemma 2.5 [44] *If $\rho \in U_\infty$, $(WPAA(\mathbb{R}, X, \rho), \|\cdot\|_{WPAA(\mathbb{R}, X, \rho)})$ is a Banach space with the supremum norm given by*

$$\|f\|_{WPAA(\mathbb{R}, X, \rho)} = \sup_{t \in \mathbb{R}} \|f(t)\|.$$

The fractional calculus was first anticipated by Leibnitz, was one of the founders of standard calculus, in a letterwritten in 1695. This calculus involves different definitions of the fractional operators as well as the Riemann-Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald-Letnikov fractional derivative. The fractional calculus has gained considerable importance during the past decades mainly due to its applications in diverse fields of science and engineering. For the purpose of this paper the Riemann-Liouville's definition of fractional derivative will be used. In the following, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.7 [70] The fractional integral of order $\alpha > 0$ with the lower limit t_0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f(s) \mathrm{d}s, \quad t > t_0, \quad \alpha > 0,$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the Gamma function.

Definition 2.8 [70] Riemann-Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (t-s)^{-\alpha} f(s)\mathrm{d}s, \quad t > t_0, \quad n-1 < \alpha < n.$$

The first and maybe the most important property of Riemann-Liouville fractional derivative is that for $t > t_0$ and $\alpha > 0$, one has $D_t^\alpha(I^\alpha f(t)) = f(t)$ which means that Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order α.

3 Existence and uniqueness of solutions

To study the existence and uniqueness of solutions to Eq. (1.1), we require the following assumptions.

(H_1) The operator A is the infinitesimal generator of an analytic semigroup $Q(t)$ satisfying

$$\|Q(t)\|_{L(X)} \le M' \exp(-\delta' t), \quad \text{for } t \ge 0;$$

(H_2) $N = \sup_{t \in \mathbb{R}} \|f(t, A^{-q}x(t))\| < \infty$, and there exists constants $\eta \in (0, 1]$, $L > 0$ such that

$$\|f(t,x) - f(\bar{t}, \bar{x})\| \le L \left(|t-\bar{t}|^\eta + \|x-\bar{x}\|_q\right), \tag{3.1}$$

for $(t, x), (\bar{t}, \bar{x}) \in \mathbb{R} \times X_q$;

Lemma 3.1 [9] *If g satisfies a uniform Hölder condition with exponent $\beta \in (0, 1]$, then the unique solution of the Cauchy problem*

$$D_t^\alpha x(t) = Ax(t) + g(t), \quad t \in I, \quad x(t_0) = x_0, \tag{3.2}$$

is given by

$$x(t) = \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) x_0 d\theta$$
$$+ \alpha \int_{t_0}^{t} \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) Q((t-\eta)^\alpha \theta) g(\eta) d\theta d\eta,$$

where $\zeta_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$. *The Laplace transform of ζ_α is given by* [71]

$$\int_0^\infty e^{-pt}\zeta_\alpha(t) = F_\alpha(p) = \sum_{j=0}^\infty \frac{(-p)^j}{\Gamma(1+\alpha j)}, \quad 0 < \alpha < 1. \tag{3.3}$$

By a classical solution to Eq. (1.1), we mean a function x with values in X such that x is continuous on $[t_0, T]$ and $x(t) \in D(A)$, moreover $\mathrm{D}_t^\alpha x(t)$ exists, continuous on (t_0, T) and x satisfies Eq. (1.1) on (t_0, T).

Theorem 3.1 *Let* (H_1) *and* (H_2) *be satisfied. Then Eq.* (1.1) *admits a unique solution for L sufficiently small.*

Proof Define $F : C(\mathbb{R}, X) \to C(\mathbb{R}, X)$ by

$$\begin{aligned}(Fx)(t) = &\int_0^\infty \zeta_\alpha(\theta) A^q Q((t-t_0)^\alpha\theta)x_0 \mathrm{d}\theta \\ &+ \alpha \int_{t_0}^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) A^q Q((t-\eta)^\alpha\theta) f(\eta, A^{-q}x(\eta))\mathrm{d}\theta \mathrm{d}\eta.\end{aligned}$$

Firstly, we show that F is well defined. □

By Lemma 2.1 (3), one has

$$\begin{aligned}\|Fx(t)\| \le M_q\|x_0\| &\int_0^\infty \zeta_\alpha(\theta) t^q e^{-\delta\theta(t-t_0)^\alpha} \mathrm{d}\theta \\ &+ \alpha N M_q \int_{t_0}^t \int_0^\infty \theta^{1-q}\zeta_\alpha(\theta)(t-\eta)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-\eta)^\alpha} \mathrm{d}\theta \mathrm{d}\eta.\end{aligned}$$

By the properties of the probability density function ζ_α, we conclude that Fx exists.

Secondly, we show that the operator F has a unique fixed point in $C(\mathbb{R}, X)$.

Taking $t \in I$ and $x_1(t), x_2(t) \in C(\mathbb{R}, X)$, using Lemma 2.1 (3) and (H_2) one gets

$$\begin{aligned}\|Fx_1(t) - Fx_2(t)\| \le &L\alpha M_q\|x_1(t) \\ &- x_2(t)\|_\infty \int_{t_0}^t \int_0^\infty \theta^{1-q}\zeta_\alpha(\theta)(t-\eta)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-\eta)^\alpha} \mathrm{d}\theta \mathrm{d}\eta.\end{aligned}$$

Set $s = t - \eta$, one obtains

$$\|Fx_1(t) - Fx_2(t)\|_\infty \leq L\alpha M_q \|x_1(t) - x_2(t)\|_\infty \int_{t_0}^{t}\int_0^\infty \theta^{1-q}\zeta_\alpha(\theta)s^{-\alpha q+\alpha-1}e^{-\delta\theta s^\alpha}\mathrm{d}\theta\mathrm{d}s.$$

It is known from above that the double integral in the right-hand side of the inequality exists, then we choose L sufficiently small, thus F is a strict contraction. By the contraction mapping theorem there exists $x(t) \in C(\mathbb{R}, X)$ such that

$$x(t) = \int_0^\infty \zeta_\alpha(\theta)A^q Q((t-t_0)^\alpha\theta)x_0\mathrm{d}\theta + \alpha\int_{t_0}^{t}\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta)f(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta. \quad (3.4)$$

Since A^q is closed, applying A^{-q} on both sides of (3.4), one gets

$$A^{-q}x(t) = \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)x_0\mathrm{d}\theta + \alpha\int_{t_0}^{t}\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)Q((t-\eta)^\alpha\theta)f(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta. \quad (3.5)$$

Thirdly, we show that the solution x of (3.5) is Hölder continuous.
In fact, let

$$x_1(t) = \int_0^\infty \zeta_\alpha(\theta)A^q Q(t^\alpha\theta)x_0\mathrm{d}\theta,$$

$$x_2(t) = \alpha\int_{t_0}^{t}\int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta)f(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta.$$

Notice that for each $x_0 \in X$,

$$\int_0^\infty\int_t^{t+h} \zeta_\alpha \frac{\mathrm{d}}{\mathrm{d}\eta}A^q Q(\eta^\alpha\theta)x_0\mathrm{d}\eta\mathrm{d}\theta = \int_0^\infty \zeta_\alpha A^q[Q((t+h)^\alpha\theta) - Q(t^\alpha\theta)]x_0\mathrm{d}\theta = \int_0^\infty\int_t^{t+h} \alpha\zeta_\alpha\eta^{\alpha-1}A^{q+1}Q(\eta^\alpha\theta)x_0\mathrm{d}\eta\mathrm{d}\theta.$$

Thus

$$\|x_1(t+h)-x_1(t)\| = \left\| \int_0^\infty \int_t^{t+h} \alpha\zeta_\alpha(\theta)\eta^{\alpha-1}A^{q+1}Q(\eta^\alpha\theta)x_0 \mathrm{d}\eta\mathrm{d}\theta \right\|$$
$$\leq \alpha\|A^{q+1}x_0\| \int_t^{t+h}\int_0^\infty \zeta_\alpha(\theta)\eta^{\alpha-1}e^{-\delta(\eta^\alpha\theta)}\mathrm{d}\eta\mathrm{d}\theta. \quad (3.6)$$

On the other hand, by Lemma 2.1 (4), for every $h>0$, β satisfying $0<\beta<1-q$, one has

$$\|(Q(h)-I)A^qQ(t-\eta)\|_{L(X)} = C_\beta h^\beta\|A^{q+\beta}Q(t-\eta)\|_{L(X)}. \quad (3.7)$$

Also for $h\geq 0$, one can write

$$\begin{aligned}&\|Q((t+h-\eta)^\alpha\theta)\|_{L(X)}\\ &\quad = \|Q((t+h-\eta)^\alpha\theta-(t-\eta)^\alpha\theta^*-h^\alpha\theta^*)Q(h^\alpha\theta^*)Q((t-\eta)^\alpha\theta^*)\|_{L(X)}\\ &\quad \leq M^*\|Q(h^\alpha\theta^*)Q((t-\eta)^\alpha\theta^*)\|_{L(X)}. \end{aligned} \quad (3.8)$$

where $\theta^*=\frac{\theta}{2}$ and M^* is a constant. Using (3.7), (3.8) and Lemma 2.1 (3), one gets

$$\begin{aligned}&\|x_2(t+h)-x_2(t)\|\\ &\quad \leq \alpha\Bigg(\left\|M^*\int_{t_0}^{t}\int_0^\infty \theta(\lambda^{\alpha-1}-\mu^{\alpha-1})\zeta_\alpha(\theta)(Q(h^\alpha\theta^*)-I)A^qQ(\mu^\alpha\theta^*)f\mathrm{d}\theta\mathrm{d}\eta\right\|\\ &\qquad +\left\|\int_t^{t+h}\int_0^\infty \theta\lambda^{\alpha-1}\zeta_\alpha(\theta)A^qQ(\lambda^\alpha\theta)f\mathrm{d}\theta\mathrm{d}\eta\right\|\Bigg)\\ &\quad \leq \alpha N\Bigg(M^*C_\beta h^{\alpha\beta}M_{q+\beta}\int_{t_0}^{t}\int_0^\infty \theta(\lambda^{\alpha-1}-\mu^{\alpha-1})\\ &\qquad \times\zeta_\alpha(\theta)(\theta^*)^\beta\mu^{-\alpha(q+\beta)}e^{-\delta\mu^\alpha\theta^*}(\theta^*)^{-(q+\beta)}\mathrm{d}\theta\mathrm{d}\eta\\ &\qquad +M_q\int_t^{t+h}\int_0^\infty \theta\lambda^{\alpha-1}\zeta_\alpha(\theta)\lambda^{-\alpha q}\theta^{-q}e^{-\delta\lambda^\alpha\theta}\mathrm{d}\theta\mathrm{d}\eta\Bigg), \end{aligned} \quad (3.9)$$

where

$$\lambda=t+h-\eta,\ \ \mu=t-\eta,\ \ f=f(\eta,A^{-q}x(\eta)).$$

Combine (3.6) with (3.9), one can estimate each term of the inequality separately to get

$$\begin{aligned}\|x(t+h)-x(t)\| &\le \|x_1(t+h)-x_1(t)\| \\ &\quad + \|x_2(t+h)-x_2(t)\| \le Ch^{\alpha\beta},\end{aligned}$$

which means that x is Hölder continuous. From assumption (H_2), one has

$$\begin{aligned}\|f(t, A^{-q}x(t)) - f(s, A^{-q}x(s))\| &\le L(|t-s|^{\eta} \\ &\quad + \|x(t)-x(s)\| + \|Gx(t)-Gx(s)\|). \qquad (3.10)\end{aligned}$$

Using (H_2) again, one can deduces that $t \to f(t, A^{-q}x(t))$ is Hölder continuous.

Let $x(t)$ be the solution of (3.4) and consider the equation

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + f(t, A^{-q}x(t)), \ \ t \in I, \ x(t_0) = x_0. \qquad (3.11)$$

By Lemma 3.1, one can deduce that (3.11) has a unique solution given by

$$\begin{aligned}y(t) &= \int_0^{\infty} \zeta_\alpha(\theta) Q((t-t_0)^{\alpha}\theta)x_0 \mathrm{d}\theta \\ &\quad + \alpha \int_{t_0}^{t}\int_0^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)Q((t-\eta)^{\alpha}\theta) f(\eta, A^{-q}x(\eta))\mathrm{d}\theta \mathrm{d}\eta. \qquad (3.12)\end{aligned}$$

Moreover,

$$y(t) \in D(A) \subset D(A^q), \ \text{ for all } \ t \in I.$$

Applying A^q on both sides of (3.12), one gets

$$\begin{aligned}A^q y(t) &= \int_0^{\infty} \zeta_\alpha(\theta) A^q Q((t-t_0)^{\alpha}\theta)x_0 \mathrm{d}\theta \\ &\quad + \alpha \int_{t_0}^{t}\int_0^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^{\alpha}\theta) f(\eta, A^{-q}x(\eta))\mathrm{d}\theta \mathrm{d}\eta = x(t).\end{aligned}$$

Clearly

$$y(t) = A^{-q}x(t)$$

is a solution to Eq. (1.1), the uniqueness of y follows from the uniqueness of (3.4) and (3.12). This completes the proof of the theorem.

4 Weighted pseudo-almost automorphic classical solutions

In the proof of our main result, we need the following assumptions and technical lemmas.

(H_3) $f \in PAP(X_q, X, \rho)$ satisfying (3.1).

Lemma 4.1 [37] *Let $f : \mathbb{R} \times \Omega \to X$ be uniformly almost automorphic and $x : \mathbb{R} \to \Omega$ be an almost automorphic function such that $\overline{\mathbb{R}(x)} \subset \Omega$, then $t \to f(t, x(t))$ is also almost automorphic.*

Lemma 4.2 *Let $\rho \in V_\infty$ and (H_4) be satisfied, for $x \in PAA(\mathbb{R}, X, \rho)$, $t \to f(t, A^{-q}x(t))$ is weighted pseudo-almost automorphic.*

Proof Since

$$x \in PAA(\mathbb{R}, X, \rho), \quad f \in PAA(\mathbb{R}, X_q, X, \rho),$$

hence, they can be expressed as $x = y + z$ and $f = g + \varphi$, where $y \in AA(\mathbb{R}, X)$, $g \in AA(\mathbb{R}, X, X)$ and z, φ are bounded continuous function such that

$$\lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \|z(t)\|\rho(t)\mathrm{d}t = 0, \tag{4.1}$$

$$\lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \|\varphi(t, A^{-q}x(t), G(A^{-q}x(t)))\|\rho(t)\mathrm{d}t = 0. \tag{4.2}$$

Since

$$\begin{aligned} f(t, A^{-q}x(t)) &= f(t, A^{-q}x(t)) - f(t, A^{-q}y(t)) + g(t, A^{-q}y(t)) + \varphi(t, A^{-q}y(t)) \\ &= I_1(t) + I_2(t) + I_3(t), \end{aligned}$$

where

$$I_1 = f(t, A^{-q}x(t)) - f(t, A^{-q}y(t)), \quad I_2 = g(t, A^{-q}y(t)), \quad I_3 = \varphi(t, A^{-q}y(t)).$$

It follows from Lemma 4.1 that $t \to f(t, A^{-q}y(t))$ is almost automorphic. According to (H_3) and (4.1), (4.2), one has

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \|I_1(t) + I_3(t)\|\rho(t)\mathrm{d}t \\ &\le \lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} (\|I_1(t)\| + \|I_3(t)\|)\rho(t)\mathrm{d}t = 0. \end{aligned}$$

□

Lemma 4.3 *If $g : \mathbb{R} \to X$ is weighted pseudo-almost automorphic and locally Hölder continuous, then there exists one and only one weighted pseudo-almost automorphic classical solution over $\mathbb{R}$ to Eq. (3.2) given by*

$$x(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)Q((t-\eta)^\alpha\theta)g(\eta)d\theta d\eta.$$

For the proof we use the same technique which appear in Zaidman [72].

Theorem 4.1 *Let $\rho \in V_\infty$ and (H_1) and (H_3) be satisfied. Then Eq. (1.1) has a unique weighted pseudo-almost automorphic ciassical solution for L sufficiently small.*

Proof From Pazy [67], it is clear that if f is Hölder continuous and A generates an analytic semigroup, then the mild solution to Eq. (1.1) in fact is a classical solution. Let $x \in PAA(\mathbb{R}, X, \rho)$, using a standard properties of the weighted pseudo almost automorphy, one has

$$N = \sup_{t\in\mathbb{R}} \|f(t, A^{-q}x(t))\| < \infty,$$

thus (H_3) implies (H_2). According to Theorem 3.1, a solution to Eq. (1.1) can be formally represented by (3.12). When A generates a semigroup with negative exponent, one deduces that if $x(\cdot)$ is a bounded mild solution to Eq. (1.1), then taking the limit as $t_0 \to -\infty$ on the right-hand side of (3.12) and using (3.3), one obtains

$$A^{-q}x(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)Q((t-\eta)^\alpha\theta)f(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta. \quad (4.3)$$

Conversely, if $x(\cdot)$ is a bounded continuous function and (4.3) is verified, then $x(\cdot)$ is a mild solution to Eq. (1.1). Define

$$Tx(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta)f(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta. \quad (4.4)$$

Firstly, we show that T is well defined. □

By Lemma 2.1 (3), one has

$$\|Tx(t)\| \leq \alpha N M_q \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q}\zeta_\alpha(\theta)(t-\eta)^{-\alpha q+\alpha-1}e^{-\delta\theta(t-\eta)^\alpha}\mathrm{d}\theta\mathrm{d}\eta.$$

By the properties of the probability density function ζ_α and the definition of the gamma function we conclude that Tx exists.

Secondly, we show the operator T maps $PAA(\mathbb{R}, X, \rho)$ into itself. Furthermore, T has a unique fixed point in $PAP(X, \rho)$.

It follows from Lemma 4.2 that for $x \in PAA(\mathbb{R}, X, \rho)$, $t \to f(t, A^{-q}x(t))$ is weighted pseudo-almost automorphic. Hence, it can be expressed as $f = g + \varphi$, where $g \in AA(\mathbb{R}, X)$ and φ is bounded continuous function such that

$$\lim_{T\to\infty} \frac{1}{m(T,\rho)} \int_{-T}^{T} \|\varphi(t, A^{-q}x(t))\|\rho(t)\mathrm{d}t = 0.$$

Since $g \in AA(\mathbb{R}, X)$, then for every sequence of real numbers $\{s'_n\}_{n=1}^{\infty}$, one can extract a subsequence $\{s_n\}_{n=1}^{\infty}$ and a function $\tilde{g}$ such that

$$\|g(t+s_n, A^{-q}x(t+s_n)) - \tilde{g}(t, A^{-q}x(t))\| < \varepsilon,$$

and

$$\|\tilde{g}(t-s_n, A^{-q}x(t-s_n)) - g(t, A^{-q}x(t))\| < \varepsilon.$$

Therefore, the map T defined by (4.4) satisfies

$$\begin{aligned} Tx(t) &= \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta) g(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta \\ &\quad + \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta) \varphi(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta \\ &= T_g x(t) + T_\varphi x(t). \end{aligned}$$

Let

$$T_{\tilde{g}}x(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta) g(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta.$$

Thus, one has

$$\begin{aligned} &\|T_g x(t+s_n) - T_{\tilde{g}}x(t)\| \\ &= \Bigg\| \alpha \int_{-\infty}^{t+s_n} \int_{0}^{\infty} \theta(t+s_n-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t+s_n-\eta)^\alpha\theta) g(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta \\ &\quad - \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-\eta)^\alpha\theta) \tilde{g}(\eta, A^{-q}x(\eta))\mathrm{d}\theta\mathrm{d}\eta \Bigg\| \end{aligned}$$

$$= \left\| \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-\eta)^\alpha \theta)(g(\eta+s_n, A^{-q}x(\eta+s_n)) \right.$$

$$\left. - \tilde{g}(\eta, A^{-q}x(\eta))) \mathrm{d}\theta \mathrm{d}\eta \right\|$$

$$\leq \varepsilon \alpha M_q \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_\alpha(\theta)(t-s)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-s)^\alpha} d\theta ds.$$

Proceeding as previously, one can show that

$$\|T_{\tilde{g}}x(t-s_n) - T_g x(t)\| \leq \varepsilon \alpha M_q \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_\alpha(\theta)(t-s)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-s)^\alpha} d\theta ds,$$

that is $T_g x(t) \in AA(\mathbb{R}, X)$. On the other hand

$$\lim_{T\to\infty} \frac{1}{m(T,\rho)} \int_{-T}^{T} \|T_\varphi x(t)\| \rho(t) \mathrm{d}t$$

$$= \lim_{T\to\infty} \frac{\alpha}{m(T,\rho)} \int_{-T}^{T} \left\| \int_{-\infty}^{t} \int_{0}^{\infty} \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-\eta)^\alpha \theta) \right.$$

$$\left. \times \varphi(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta \right\| \rho(t) \mathrm{d}t$$

$$\leq \lim_{T\to\infty} \frac{\alpha M_q}{m(T,\rho)} \int_{-T}^{T} \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_\alpha(\theta)(t-\eta)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-\eta)^\alpha} \|$$

$$\times \varphi(\eta, A^{-q}x(\eta)) \| \mathrm{d}\theta \mathrm{d}\eta \rho(t) \mathrm{d}t$$

$$= \alpha M_q \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_\alpha(\theta)(t-\eta)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-\eta)^\alpha} \mathrm{d}\theta \mathrm{d}\eta$$

$$\times \lim_{T\to\infty} \frac{1}{m(T,\rho)} \int_{-T}^{T} \|\varphi(\eta, A^{-q}x(\eta))\| \rho(t) \mathrm{d}t.$$

Then one has

$$\lim_{T\to\infty} \frac{1}{m(T,\rho)} \int_{-T}^{T} \|T_\varphi x(t)\| \rho(t) \mathrm{d}t = 0,$$

which means $T_\varphi x(t) \in PAP_0(\mathbb{R}, X, \rho)$.

Let $x_1(t), x_2(t) \in PAA(\mathbb{R}, X, \rho)$, using Lemma 2.1 (3) and assumption (H_3) one gets

$$\|Tx_1(t) - Tx_2(t)\| \\ \leq L\alpha M_q \|x_1(t) - x_2(t)\|_\infty \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_\alpha(\theta)(t-\eta)^{-\alpha q+\alpha-1} e^{-\delta\theta(t-\eta)^\alpha} \mathrm{d}\theta \mathrm{d}\eta.$$

Set $s = t - \eta$, one obtains

$$\|Tx_1(t) - Tx_2(t)\|_\infty \\ \leq L\alpha M_q \|x_1(t) - x_2(t)\|_\infty \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_\alpha(\theta) s^{-\alpha q+\alpha-1} e^{-\delta\theta s^\alpha} \mathrm{d}\theta \mathrm{d}s.$$

It is known from above that the double integral in the right-hand side of the inequality exists, then we choose L sufficiently small, thus T is a strict contraction. By the contraction mapping theorem there exists $x(t) \in PAP(\mathbb{R}, X, \rho)$ such that

$$x(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-\eta)^\alpha \theta) f(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta. \quad (4.5)$$

Since A^q is closed, applying A^{-q} on both sides of (4.5), one gets

$$A^{-q}x(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-\eta)^{\alpha-1} \zeta_\alpha(\theta) Q((t-\eta)^\alpha \theta) f(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta.$$

Finally, we show that the solution x of (4.5) is Hölder continuous.
Using (3.7), (3.8) and Lemma 2.1 (3), one gets

$$\|x(t+h) - x(t)\| \\ \leq \alpha \Bigg(\Bigg\| M^* \int_{-\infty}^{t} \int_{0}^{\infty} \theta(\lambda^{\alpha-1} - \mu^{\alpha-1}) \\ \times \zeta_\alpha(\theta)(Q(h^\alpha \theta^*) - I) A^q Q(\mu^\alpha \theta^*) f(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta \Bigg\| \\ + \Bigg\| \int_{t}^{t+h} \int_{0}^{\infty} \theta \lambda^{\alpha-1} \zeta_\alpha(\theta) A^q Q(\lambda^\alpha \theta) f(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta \Bigg\| \Bigg) \\ \leq \alpha M^* C_\beta h^{\alpha\beta} N M_{q+\beta} \int_{-\infty}^{t} \int_{0}^{\infty} \theta(\lambda^{\alpha-1} - \mu^{\alpha-1})$$

$$\times \zeta_\alpha(\theta)(\theta^*)^\beta \mu^{-\alpha(q+\beta)}(\theta^*)^{-(q+\beta)} e^{-\delta\mu^\alpha\theta^*} \mathrm{d}\theta \mathrm{d}\eta$$
$$+ \alpha N M_q \int_t^{t+h} \int_0^\infty \theta\lambda^{\alpha-1}\zeta_\alpha \lambda^{-\alpha q}\theta^{-q} e^{-\delta\lambda^\alpha\theta} \mathrm{d}\theta \mathrm{d}\eta,$$

where

$$\lambda = t + h - \eta, \ \ \mu = t - \eta.$$

One can estimate each term of the inequality separately to get

$$\|x(t+h) - x(t)\| \leq C h^{\alpha\beta},$$

which means that x is Hölder continuous. Combine (3.10) with (H_3), one can deduce that $t \to f(t, A^{-q}x(t))$ is Hölder continuous. Let x be the solution of (4.5) and consider the equation

$$\mathrm{D}_t^\alpha x(t) = Ax(t) + f(t, A^{-q}x(t)), \ \ t \in I. \tag{4.6}$$

By Lemma 4.3, Eq. (4.6) has a unique weighted pseudo almost automorphic solution given by

$$y(t) = \alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) Q((t-\eta)^\alpha\theta) f(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta. \tag{4.7}$$

Moreover, $y(t) \in D(A) \subset D(A^q)$ for all $t \in \mathbb{R}$. Applying A^q on both sides of (4.7), one gets

$$A^q y(t) = \alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta) Q((t-\eta)^\alpha\theta) f(\eta, A^{-q}x(\eta)) \mathrm{d}\theta \mathrm{d}\eta = x(t).$$

Clearly $y(t) = A^{-q}x(t)$ is a solution to Eq. (1.1), the uniqueness of $y(t)$ follows from the uniqueness of the solution of (4.5) and (4.7). This completes the proof of the theorem.

5 Optimal mild solutions

Before starting our main results in this subsection, we recall the definition of the mild solution to Eq. (1.1).

Definition 5.1 A continuous function $x : I \to X$ satisfying the integral equation

$$x(t) = \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) x_0 \mathrm{d}\theta + \alpha \int_{t_0}^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) Q((t-\eta)^\alpha \theta) f(\eta, x(\eta)) \mathrm{d}\theta \mathrm{d}\eta. \qquad (5.1)$$

is called a mild solution to Eq. (1.1).

As in [37], we consider in X the Eq. (1.1) with the following assumptions

(H_4) $A : D(A) \subset X \to X$ generates a C_0 semigroup $Q(t)$ satisfying

$$\sup_{t \in \mathbb{R}^+} \|Q(t)\|_{L(X)} < \infty;$$

(H_5) $f : \mathbb{R} \times X_q \to X$ is a nontrivial strongly continuous function satisfying (3.1), moreover f is convex in $x \in X_q$.

(H_6) $\lim_{n\to\infty} \|x_n(t) - x(t)\| = 0$ implies $\lim_{n\to\infty} \|x_n(t) - x(t)\|_q = 0$, for $x_n(t), x(t) \in X$.

Denote by Ω_f the set of all mild solutions $x(t)$ to Eq. (1.1) which are bounded over $\mathbb{R}$, that is

$$\mu(x) = \sup_{t \in \mathbb{R}} \|x(t)\| < \infty.$$

Assume here that $\Omega_f \neq \emptyset$, and recall the following

Definition 5.2 A bounded mild solution $\tilde{x}(t)$ to Eq. (1.1) is called an optimal mild solution to Eq. (1.1) if

$$\mu(\tilde{x}) \equiv \mu^* = \inf_{x \in \Omega_f} \mu(x).$$

Our proof is based on the following lemma.

Lemma 5.1 [73] *If K is a nonempty convex and closed subset of a uniformly convex Banach space X and $v \notin K$, then there exists a unique $k_0 \in K$ such that*

$$\|v - k_0\| = \inf_{k \in K} \|v - k\|.$$

Theorem 5.1 *Assume that $\Omega_f \neq \emptyset$ and (H_4), (H_5), (H_6) hold, then Eq. (1.1) has a unique optimal mild solution.*

Proof It suffices to prove that Ω_f is a convex and closed set because the trivial solution $0 \notin \Omega_f$, then we use Lemma 5.1 to deduce the uniqueness of the optimal mild solution. For the convexity of Ω_f, consider two distinct bounded mild solutions $x_1(t)$, $x_2(t)$ and a real number $0 \le \lambda \le 1$, let

$$x(t) = \lambda x_1(t) + (1-\lambda)x_2(t),\ \ t \in \mathbb{R}.$$

For every $t_0 \in \mathbb{R},\ x(t)$ is continuous and has the integral representation

$$x(t) = T(t-t_0)x(t_0) + \int_{t_0}^{t} S(t-\eta)f(\eta, x(\eta))\mathrm{d}\eta,\ \ t > t_0, \tag{5.2}$$

where

$$T(t) = \int_0^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)\mathrm{d}\theta,\ \ S(t) = \alpha\int_0^\infty \theta t^{\alpha-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)\mathrm{d}\eta.$$

One has

$$x(t_0) = \lambda x_1(t_0) + (1-\lambda)x_2(t_0)$$

and $f(t,x)$ is convex in x, then $x(t)$ is a mild solution to Eq. (1.1). Note that $x(t)$ is bounded over $\mathbb{R}$ since

$$\mu(x) = \sup_{t\in\mathbb{R}} \|x(t)\| \le \lambda\mu(x_1) + (1-\lambda)\mu(x_2) < \infty.$$

One concludes that $x(t) \in \Omega_f$. Now we show that Ω_f is closed, let a sequence $x_n \in \Omega_f$ such that

$$\lim_{n\to\infty} x_n(t) = x(t),\ \ t \in \mathbb{R}.$$

For all $t_0 \in \mathbb{R}$ and $t \ge t_0$ one has

$$x_n(t) = T(t-t_0)x_n(t_0) + \int_{t_0}^{t} S(t-\eta)f(\eta, x_n(\eta))\mathrm{d}\eta.$$

It is clearly that $T(t-t_0)$ and $S(t-\eta)$ are continuous operators, then for every fixed t and t_0 with $t \ge t_0$, one has

$$\begin{aligned}\lim_{n\to\infty} T(t-t_0)x_n(t_0) &= \lim_{n\to\infty}\int_0^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)x_n(t_0)\mathrm{d}\theta\\ &= \int_0^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)\mathrm{d}\theta \lim_{n\to\infty} x_n(t_0)\\ &= T(t-t_0)\lim_{n\to\infty} x_n(t_0) = T(t-t_0)x(t_0).\end{aligned}$$

Similarly one has

$$\lim_{n\to\infty}\int_{t_0}^{t} S(t-\eta)f(\eta,x_n(\eta))\mathrm{d}\eta = \int_{t_0}^{t} S(t-\eta)\lim_{n\to\infty} f(\eta,x_n(\eta))\mathrm{d}\eta$$
$$= \int_{t_0}^{t} S(t-\eta)f(\eta,x(\eta))\mathrm{d}\eta.$$

Then one deduces that

$$x(t) = T(t-t_0)x(t_0) + \int_{t_0}^{t} S(t-\eta)f(\eta,x(\eta))\mathrm{d}\eta,$$

for all $t_0 \in \mathbb{R},\ t > t_0$, which means that $x(t)$ is a mild solution to Eq. (1.1). □

Finally we show that $x(t)$ is bounded over $\mathbb{R}$. One can write (6.2) as

$$x(t) = T(t-t_0)x(t_0) + \int_{t_0}^{t} S(t-\eta)f(\eta,x(\eta))\mathrm{d}\eta - x_n(t) + x_n(t)$$
$$= T(t-t_0)[x(t_0)-x_n(t_0)] + \int_{t_0}^{t} S(t-\eta)[f(\eta,x(\eta)) - f(\eta,x_n(\eta))]\mathrm{d}\eta + x_n(t),$$

for every $n = 1, 2, \ldots$, and every $t_0 \in \mathbb{R}$ such that $t > t_0$.

Let

$$M = \sup_{t\in\mathbb{R}^+} \|Q(t^\alpha\theta)\| < \infty,$$

since

$$\int_0^\infty \zeta_\alpha(\theta)d\theta = 1,$$

then $\|T(t)\| \leq M$, again since

$$\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1,$$

then

$$\|S(t)\| \leq \alpha|t|^{\alpha-1} \sup_{t\in\mathbb{R}^+} \|Q(t^\alpha\theta)\| \leq M\alpha|t|^{\alpha-1}.$$

Combine (H_5) with (H_6), one has

$$\|x(t)\| \leq M\|x(t_0) - x_n(t_0)\| + \alpha ML \int_{t_0}^{t} |t-\eta|^{\alpha-1} \|x(\eta) - x_n(\eta)\|_q \mathrm{d}\eta + \|x_n(t)\|.$$

Choose n large enough and using (H_7), for every $\varepsilon > 0$ one gets

$$\|x(t)\| \leq M\varepsilon + \alpha ML\varepsilon \int_{t_0}^{t} |t-\eta|^{\alpha-1} \mathrm{d}\eta + \mu(x_n),$$

then one has

$$\mu(x) \leq \varepsilon_1 + \varepsilon_2 + \mu(x_n) < \infty.$$

Thus $x \in \Omega_f$. This completes the proof of the theorem.

6 Applications

In this section, as example and applications, we study the weighted pseudo-almost automorphic classical solutions and optimal mild solutions for a fractional reaction–diffusion equation to illustrate the practical usefulness of the results that we establish in the paper.

A fractional reaction–diffusion equation is derived from a continuous time random walk model when the transport is dispersive. In the recent years, there has been a great deal of interest in fractional reaction–diffusion systems which from one side exhibit selforganization phenomena and from the other side introduce a new parameter to these systems, which is a fractional derivative index, and it gives a great degree of freedom for diversity of selforganization phenomena and new nonlinear effects depending on the order of time-space fractional derivatives. From a mathematics point of view they also offer a rich and promising area of research. The most important advantage of using fractional differential equations is their nonlocal property. This indicates that the next state of a system depends not only upon its current state but also upon all of its previous states. In recent years, the fractional reaction–diffusion equation has received the applications in systems biology [25,26], chemistry, and biochemistry applications [27]. A strong motivation for studying and investigating the solution and the properties for fractional diffusion equations comes from the fact that they describe efficiently anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [17,28–33] and references therein), fractional random walk, etc.

Consider the following fractional reaction–diffusion equations

$$\begin{aligned} &\partial_t^{\alpha} u(t,x) = \partial_x^2 u(t,x) + F(t, u(t,x)), \;\; t \in \mathbb{R}, \; x \in [0,\pi], \\ &u(t,0) = u(t,\pi) = 0, \;\; t \in \mathbb{R}, \end{aligned} \tag{6.1}$$

where

$$F(t, u(t, x)) = \beta \sin(\partial_x u(t, x)) \left[\sin\left(\frac{1}{2 + \cos(t^{1/2}) + \cos(\sqrt{2}t^{1/2})} \right) + t^{1/2} e^{-|t|} \right].$$

Let $u(s, x) = \varphi(s, x)$, $\varphi(\cdot, x) \in C([0, T], \mathbb{R}^1)$, $\varphi(s, \cdot) \in L^2([0, \pi], \mathbb{R})$, $s \in [0, T]$, $x \in [0, \pi]$. Denote $X = L^2([0, \pi], \mathbb{R})$ and define $A : D(A) \subset X \to X$ given by $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$D(A) = \Big\{ u(\cdot) \in X : u'' \in X, u' \in X \text{ is absolutely continuous on } [0, \pi], u(0) = u(\pi) = 0 \Big\}.$$

Taking $\alpha = 1/2$, that is $X_{1/2} = (D(A^{1/2}), \|\cdot\|_{1/2})$. In the following, we give some known results for the operators A and $A^{1/2}$.

It is well known that A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\|T(t)\| \leq e^{-t}$ for $t \geq 0$. Let $u \in D(A)$ and $\lambda \in \mathbb{R}$, such that $Au = -u'' = \lambda u$, that is,

$$u'' + \lambda u = 0. \tag{6.2}$$

Thus one has $< Au, u >=< \lambda u, u >$, that is $< -u'', u >= \|u'\|^2_{L^2([0,\pi],\mathbb{R})} = \lambda \|u\|^2_{L^2([0,\pi],\mathbb{R})}$. The solutions of (6.2) have the form

$$u(x) = C \cos(\sqrt{\lambda} x) + D \sin(\sqrt{\lambda} x).$$

From $u(0) = u(\pi)$, it follows that $C = 0$ and $\sqrt{\lambda} = n$, $n \in N$. Put $\lambda_n = n^2$, the solutions of Eq. (6.2) are

$$u_n(x) = D \sin(\sqrt{\lambda_n} x),\ n \in N.$$

According to $< u_n, u_m >= 0$, for $n \neq m$ and $< u_n, u_n >= 1$, one has $D = \sqrt{2}$ and

$$u_n(x) = \sqrt{2} \sin(\sqrt{\lambda_n} x).$$

For $u \in D(A)$, there exists a sequence of reals (α_n) such that

$$u(x) = \sum_{n\in N} \alpha_n u_n(x),$$
$$\sum_{n\in N} \alpha_n^2 < +\infty, \quad \sum_{n\in N} \lambda_n^2 \alpha_n^2 < +\infty.$$

Thus one has

$$A^{1/2} u(x) = \sum_{n\in N} \sqrt{\lambda_n} \alpha_n u_n(x).$$

with $u \in D(A^{1/2})$, that is,

$$\sum_{n\in N} \alpha_n^2 < +\infty, \ \sum_{n\in N} \lambda_n \alpha_n^2 < +\infty.$$

Let

$$\begin{aligned} f(t,\varphi)(\eta) &= \beta \sin(\varphi_x(\eta)) \left[\sin\left(\frac{1}{2+\cos(t^{1/2})+\cos(\sqrt{2}t^{1/2})} \right) + t^{1/2}e^{-|t|} \right] \\ &= h(t)g(\varphi'), \end{aligned}$$

for each $t \in \mathbb{R}$ and $u \in X_{\frac{1}{2}}$, where

$$h(t) = \beta \left[\sin\left(\frac{1}{2+\cos(t^{1/2})+\cos(\sqrt{2}t^{1/2})} \right) + t^{1/2}e^{-|t|} \right] \text{ and } g(\varphi') = \sin(\varphi_x(\eta)).$$

Note that $h(t)$ is weighted pseudo almost automorphic in $\mathbb{R}$ satisfying

$$|h(t) - h(s)| = \beta(1+\sqrt{2})|t-s|^{1/2}, \tag{6.3}$$

and g is Lipschitz continuous on X that is

$$\|g(\varphi_1') - g(\varphi_2')\| = \|\varphi_1' - \varphi_2'\|_{L^2([0,\pi],\mathbb{R})}. \tag{6.4}$$

We show now that f satisfies the hypothesis (H_2). In fact, for $t_1, t_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in X_{1/2}$, one has

$$\begin{aligned} \|f(t_1,\varphi_1) - f(t_2,\varphi_2)\| &= \|h(t_1)g(\varphi_1') - h(t_2)g(\varphi_2')\| \\ &= \|[h(t_1) - h(t_2)]g(\varphi_1') + h(t_2)[g(\varphi_1') - g(\varphi_2')]\| \\ &\le \|h(t_1) - h(t_2)\| + |h(t_2)|\|\varphi_1' - \varphi_2'\|. \end{aligned} \tag{6.5}$$

Since h is weighted pseudo almost automorphic, there exists $K > 0$, such that $|h(t_2)| \le K$, (Note that $K \le \beta(1+\sqrt{2})$). Therefore, from (6.3), (6.4), (6.5), and the fact that $g(\varphi')$ is Lipschitz on $X_{1/2}$ (see for instance [74]), one has

$$\begin{aligned} \|f(t_1,\varphi_1) - f(t_2,\varphi_2)\| &= \|h(t_1)g(\varphi_1') - h(t_2)g(\varphi_2')\| \\ &\le \beta(1+\sqrt{2})|t_1 - t_2| + K|\|\varphi_1 - \varphi_2\|_{1/2} \\ &\le \beta(1+\sqrt{2})(|t_1 - t_2| + |\|\varphi_1 - \varphi_2\|_{1/2}). \end{aligned}$$

Therefore, f satisfies the hypothesis (H_2) with $L = \beta(1+\sqrt{2})$.

From Theorems 4.1 and 5.1, it follows that the following proposition holds.

Proposition 6.1 *Equation* (6.1) *has a unique weighted pseudo-almost automorphic classical solution and a unique optimal mild solution for* β *sufficiently small.*

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References

1. D. Benson, S. Wheatcraft, M. Meerschaert, Application of fractional advection–dispersion equation. Water Resour. Res. **36**, 1403–1412 (2000)
2. R. Metzler, J. Klafter, Accelerating Brownian motion: a fractional dynamics approach to fast diffusion. Europhys. Lett. **51**, 492–498 (2000)
3. J. Machado, Fractional derivatives: probability interpretation and frequency response of rational approximations. Commun. Nonlinear Sci. Numer. Simul. **14**, 3492–3497 (2009)
4. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. J. Contam. Hydrol. **85**, 53–71 (2000)
5. I. Podlubny, *Fractional Differential Equations* (Academic Press, New York, 1999)
6. V. Ahn, R. McVinisch, Fractional differential equations driven by Lévy noise. J. Appl. Math. Stoch. Anal. **16**, 97–119 (2003)
7. D. Benson, *The Fractional Advection–Dispersion Equation*, Ph.D. Thesis (University of Nevada, Reno, 1998)
8. R. Schumer, D. Benson, Eulerian derivative of the fractional advection–dispersion equation. J. Contam. Hydrol. **48**, 69–88 (2001)
9. M. Borai, Some probability densities and fundamental solutions of fractional evolutions equations. Chaos Solitons Fract. **14**, 433–440 (2002)
10. G. Mophou, G. N'Guérékata, Existence of mild solution for some fractional differential equations with nonlocal conditions. Semigroup Forum **79**, 315–322 (2009)
11. V. Lahshmikantham, J. Devi, Theory of fractional differential equations in Banach spaces. Eur. J. Pure Appl. Math. **1**, 38–45 (2008)
12. B. Slepchenko, J. Schaff, Y. Choi, Numerical approach to fast reactions in reaction–diffusion systems: application to buffered calcium waves in bistable models. J. Comput. Phys. **162**, 186–189 (2000)
13. C. Vidal, A. Pascault, *Non-equilibrium Dynamics in Chemical Systems* (Wiley, New York, 1986)
14. J. Murray, *Lectures on Non-linear Differential Equation Models in Biology* (Clarenden, Oxford, 1977)
15. S. Muller, T. Plesser, B. Hess, Two-dimensional spectrophotometry of spiral wave propagation in the BelousovCZhabotinskii reaction, I: experiments and digital data representation. Physica D **24**, 71–78 (1987)
16. A. Winfree, Spiral waves of chemical activity. Science **175**, 634–642 (1972)
17. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**, 1–77 (2000)
18. R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. J. Phys. A **37**, 161–208 (2004)
19. J. Klafter, G. Zumofen, M. Shlesinger, in *The Physics of Complex Systems*, eds. by F. Mallamace, H.E. Stanley (IOS Press, Amsterdam, 1997)
20. A. Chaves, A fractional diffusion equation to describe Lévy flights. Phys. Lett. A **239**, 13–16 (1998)
21. K. Oldham, J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974)
22. R. Gorenflo, F. Mainardi, On Mittag–Leffler-type functions in functional evolution processes. J. Comput. Appl. Math. **118**, 283–299 (2000)
23. F. Mainardi, G. Pagnini, The wright functions as solutions of time-fractional diffusion equation. Appl. Math. Comput. **141**, 51–62 (2003)
24. J. Sung, E. Barkai, R.J. Silbey, S. Lee, Fractional dynamics approach to diffusion-assisted reactions in disordered media. J. Chem. Phys. **116**, 2338–2341 (2002)
25. S. Aydin, Solving time-fractional reaction–diffusion equation by reduced differential transform method. J. Comput. Eng. Technol. **3**, 19–22 (2012)
26. V. Ananthaswamy, A. Eswari, L. Rajendran, Nonlinear reaction–diffusion process in a thin membrane and Homotopy analysis method. Int. J. Autom. Control Eng. **2**, 10C18 (2013)
27. D. Olmos, A pseudo-spectral method of solution of Fisher's equation. J. Comput. Appl. Math. **193**, 219–242 (2006)

28. V. Anh, R. Mcvinish, Fractional differential equations driven by Levy noise. J. Appl. Math. Stoch. Anal. **16**, 97–119 (2003)
29. M. Mahalakshmi, G. Hariharan, K. Kannan, The wavelet methods to linear and nonlinear reaction–diffusion model arising in mathematical chemistry. J. Math. Chem. **51**, 2361–2385 (2013)
30. A. Meena, A. Eswari, L. Rajendran, Mathematical modelling of enzyme kinetics reaction mechanisms and analytical solutions of non-linear reaction equations. J. Math. Chem. **48**, 179–186 (2010)
31. L. Rajendran, R. Senthamarai, Traveling-wave solution of non-linear coupled reaction diffusion equation arising in mathematical chemistry. J. Math. Chem. **46**, 550–561 (2009)
32. K. Seki, M. Wojcik, M. Tachiya, Fractional reaction–diffusion equation. J. Chem. Phys. **119**, 2165–2174 (2003)
33. N. Khan, N. Khan, A. Ara, M. Jamil, Approximate analytical solutions of fractional reaction–diffusion equations. J. King Saud Univ. Sci. **24**, 111–118 (2012)
34. S. Bochner, Uniform convergence of monotone sequences of functions. Proc. Natl. Acad. Sci. USA **47**, 582–585 (1961)
35. S. Bochner, A new approach to almost periodicity. Proc. Natl. Acad. Sci. USA **48**, 2039–2043 (1962)
36. S. Bochner, Continuous mappings of almost automorphic and almost periodic functions. Proc. Natl. Acad. Sci. USA **52**, 907–910 (1964)
37. G. N'Guérékata, *Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces* (Kluwer Academic, Plenum Publishers, New York, 2001)
38. G. N'Guérékata, *Topics in Almost Automorphy* (Springer, New York, 2005)
39. G. Mophou, G. N'Guérékata, On some classes of almost automorphic functions and applications to fractional differential equations. Comput. Math. Appl. **59**, 1310–1317 (2010)
40. G. N'Guérékata, Existence and uniqueness of almost automorphic mild solutions of some semilinear abstract differential equations. Semigroup Forum. **69**, 80–86 (2004)
41. D. Bugajewski, T. Diagana, Almost automorphy of the convolution operator and applications to differential and functional differential equations. Nonlinear Stud. **13**, 129–140 (2006)
42. C. Cuevas, C. Lizama, Almost automorphic solutions to a class of semilinear fractional differential equations. Appl. Math. lett. **21**, 1315–1319 (2008)
43. G. N'Guérékata, Quelques remarques sur les fonctions asymptotiquement presque automorphes. Ann. SCI. Math. Québec **7**, 185–191 (1983)
44. J. Blot, G. Mophou, G. N'Guérékata, D. Pnnequin, Weighted pseudo almost automorphic functions and applications to abstract differential equations. Nonlinear Anal. Theory Methods Appl. **71**, 903–909 (2009)
45. T. Xiao, J. Liang, J. Zhang, Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces. Semigroup Forum **76**, 518–524 (2008)
46. T. Xiao, X. Zhu, J. Liang, Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications. Nonlinear Anal. Theory Methods Appl. **70**, 4079–4085 (2009)
47. C. Zhang, Integration of vector-valued pseudo almost periodic functions. Proc. Am. Math. Soc. **121**, 167–174 (1994)
48. C. Zhang, Pseudo almost periodic solutions of some differential equations. J. Math. Anal. Appl. **151**, 62–76 (1994)
49. C. Zhang, Pseudo almost periodic solutions of some differential equations II. J. Math. Anal. Appl. **192**, 543–561 (1995)
50. S. Abbas, Pseudo almost automorphic solutions of fractional order neutral differential equation. Semigroup Forum. **81**, 393–404 (2010)
51. T. Diagana, Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients. Nonlinear Anal. Theory Methods Appl. **70**, 3781–3790 (2009)
52. Y. Chang, Z. Zhao, J. Nieto, Pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to semilinear differential equations in Hilbert spaces. Rev. Mat. Complut. **24**, 421–438 (2011)
53. P. Cieutat, K. Ezzinbi, Existence, uniqueness and attractiveness of a pseudo almost automorphic solutions for some dissipative differential equations in Banach spaces. J. Math. Anal. Appl. **354**, 494–506 (2009)
54. K. Ezzinbi, S. Fatajou, G. N'Guérékata, Pseudo almost automorphic solutions to some neutral partial functional differential equations in Banach space. Nonlinear Anal. Theory Methods Appl. **70**, 1641–1647 (2009)

55. K. Ezzinbi, S. Fatajou, G. N'Guérékata, Pseudo almost automorphic solutions for dissipative differential equations in Banach spaces. J. Math. Anal. Appl. **351**, 765–772 (2009)
56. J. Liu, X. Song, Almost automorphic and weighted pseudo almost automorphic solutions of semilinear evolution equations. J. Funct. Anal. **258**, 196–207 (2010)
57. G. Mophou, Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Appl. Math. Comput. **217**, 7579–7587 (2011)
58. I. Mishra, D. Bahuguna, Weighted pseudo almost automorphic solution of an integro-differential equation with weighted Stepanov-like pseudo almost automorphic forcing term. Appl. Math. Comput. **219**, 5345–5355 (2013)
59. D. Bahuguna, S. Srvastavai, Semilinear integro-differential equations with compact semigroup. J. Appl. Math. Stoch. Anal. **11**, 507–517 (1998)
60. J. Dieudonne, *Foundation of Modern Analysis* (Academic Press, New York, 1960)
61. J. Yorke, A continuous differential equation in Hilbert space without existence. Funk. Ekvaciaj **12**, 19–21 (1970)
62. K. Balachandran, J. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations. Nonlinear Anal. Theory Methods Appl. **71**, 4471–4475 (2009)
63. B. Ahmad, S. Sivasundaram, Some existence results for fractional integrodifferential equations with nonlinear conditions. Commun. Math. Anal. **12**, 107–112 (2008)
64. G. Mophou, G. N'Guérékata, Mild solutions for semilinear fractional differential equations. Electron. J. Differ. Equ. **21**, 1–9 (2009)
65. Y. Changa, V. Kavitha, M. Arjunan, Existence and uniqueness of mild solutions to a semilinear integrodifferential equation of fractional order. Nonlinear Anal. Theory Methods Appl. **71**, 5551–5559 (2009)
66. M. Heard, S. Rankin, A semilinear parabolic integro-differential equation. J. Differ. Equ. **71**, 201–233 (1988)
67. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer, New York, 1983)
68. G. N'Guérékata, Comments on almost automorphic and almost periodic functions in Banach spaces. Far East J. Math. Sci. (FJMS) **17**(3), 337–344 (2005)
69. H. Ding, J. Liang, T. Xiao, Some properties of Stepanov-like almost automorphic functions and applications to abstract evolution equations. Appl. Anal. **88**, 1079–1091 (2009)
70. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations, North-HollandMathematics Studies, 204* (Elsevier, Amsterdam, 2006)
71. W. Feller, *An Introduction to Probability Theory and Its Applications, vol II* (Wiley, New York, 1971)
72. S. Zaidman, *Abstract Differential Equations* (Pitman Publishing, San Francisco, 1979)
73. R. Larsen, *Functional Analysis* (Decker, New York, 1973)
74. D. Henry, *Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics* (Springer, Berlin, 1981)